THE RING OF QUASIMODULAR FORMS FOR A COCOMPACT GROUP

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Abstract. We describe the additive structure of the graded ring $\widetilde{M}_*$ of quasimodular forms over any discrete and cocompact group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$. We show that this ring is never finitely generated. We calculate the exact number of new generators in each weight k. This number is constant for k sufficiently large and equals $\dim_{\mathbb{C}}(I/I \cap \widetilde{I}^2)$, where I and $\widetilde{I}$ are the ideals of modular forms and quasimodular forms, respectively, of positive weight. We show that $\widetilde{M}_*$ is contained in some finitely generated ring $\widetilde{R}_*$ of meromorphic quasimodular forms with $\dim \widetilde{R}_k = O(k^2)$, i.e. the same order of growth as $\widetilde{M}_*$.

1. Introduction

Kaneko and Zagier introduced the notion of quasimodular forms in [4]. The structure of $\widetilde{M}_*(\Gamma_1)$ (where $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$ is the classical modular group) was given in [4], in which it is proved that $\widetilde{M}_*(\Gamma_1) = \mathbb{C}[E_2, E_4, E_6]$, with E_2, E_4 and E_6 being the Eisenstein series of weights $2, 4$ and 6 respectively.

We study the ring of quasimodular forms over discrete and cocompact subgroups of $\mathrm{PSL}(2,\mathbb{R})$. In the second and third section, we derive some general properties of quasimodular forms over discrete and cocompact subgroups of $\mathrm{PSL}(2,\mathbb{R})$, following [14], [4] and [15]. In the end of the third section, we give an additive structure theorem of rings of quasimodular forms and a $\mathrm{sl}_2(\mathbb{C})$-module structure theorem for the ring of quasimodular forms. In the fourth section, we give a cocompact/no cocompact dichotomy (see Theorem 4) which characterizes cocompact modular groups in terms of their spaces of quasimodular forms of weight 2. In the fifth section, we describe our principal results which are Theorem 5 and its corollary. In Theorem 6, we describe the additive structure of the differential closure of any ring $\mathcal{M}$ generated by holomorphic or meromorphic modular forms of positive weights over

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any discrete and finite covolume subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We prove also that the differential closure of $\mathcal{M}$ is never finitely generated. In the last section, we prove the existence of quasimodular forms of weight 2 with prescribed poles. We use this result to construct finitely generated rings of meromorphic quasimodular forms with positive weights, over cocompact groups (see Theorem 10 and its corollary). In the sixth section, we give an algebraic characterization of cocompact groups, in terms of their modular forms rings (see Theorem 8).

2. General properties of quasimodular forms

In this section, we recall definitions and general properties of quasimodular forms, given by Kaneko and Zagier in [4]. We give corollaries of sevral results in [4] and new proofs for other results in this paper.

We consider a discrete and finite covolume subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$. We give the definition of modular forms, quasimodular forms, almost holomorphic modular forms and modular stacks, over the group Γ. We denote by $\mathcal{H}$ the upper half plane and by y the imaginary part of $z\in\mathcal{H}$.

Definition 1. A modular form of weight k over Γ is a holomorphic map f in $\mathcal{H}$ with moderate growth, such that :

$$(1)\qquad (cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)=f(z),\quad \forall\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma \text{ and } z\in\mathcal{H}.$$

Definition 2. A quasimodular form f of weight k and depth $\leq p$ over Γ, is a holomorphic function f in $\mathcal{H}$ with moderate growth, such that for any $z\in\mathcal{H}$, the map :

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathbb{C}\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix} & \mapsto & (cz+d)^{-k}f(\frac{az+b}{cz+d}), \end{array}$$

is a polynomial of degree $\leq p$ in $\frac{c}{cz+d}$ with functions defined on $\mathcal{H}$ as coefficients. We can write :

$$(2)\qquad (cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)=\sum_{j=0}^{p}f_j(z)\left(\frac{c}{cz+d}\right)^j,\quad \forall z\in\mathcal{H},$$

with map $f_j:\mathcal{H}\longrightarrow\mathbb{C}\quad (j=0,\cdots,p)$.

Remark 1. This definition which is different from the one given in [4] was proposed by Werner Nahm. The equivalence between this definition and the one given in [4] is a consequence of Theorem 1.

DEFINITION 3. An almost holomorphic modular form F of weight k and depth $\leq p$ over Γ is a polynomial in $\frac{1}{y}$ of degree $\leq p$ whose coefficients are holomorphic maps on $\mathcal{H}$ with moderate growth, such that (1) holds for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathcal{H}$. We can write :

$$F(z) = f_0(z) + \frac{f_1(z)}{z-\overline{z}} + \cdots + \frac{f_p(z)}{(z-\overline{z})^p},$$

with holomorphic maps (f_i), because $y = \frac{z-\overline{z}}{2i}$.

This way of writing F as polynomial in $\frac{1}{z-\overline{z}}$ is more useful for making the next calculations.

DEFINITION 4. A modular stack of weight k and depth $\leq p$ is a holomorphic map :

$$\begin{array}{rccc} E: & \mathcal{H} & \longrightarrow & \bigoplus_{l=0}^{\infty} \mathbb{C} \\ & z & \mapsto & (f_0(z), f_1(z), \cdots) \end{array}$$

with moderate growth such that the maps f_l satisfy $f_l = 0$ for $l > p$ and the functional equation :

$$(cz+d)^{-k+2l} f_l\left(\frac{az+b}{cz+d}\right) = \sum_{j \geq l} \binom{j}{l} f_j(z) \left(\frac{c}{cz+d}\right)^{j-l}. \tag{3}$$

Notation 1. We denote by $M_* = \bigoplus_{k\geq 0} M_k$ (respectively $\widetilde{M}_* = \bigoplus_{k\geq 0} \widetilde{M}_k$, $\widehat{M}_* = \bigoplus_{k\geq 0} \widehat{M}_k, \overrightarrow{M}_* = \bigoplus_{k\geq 0} \overrightarrow{M}_k$) the graded rings of modular forms (respectively quasimodular forms, almost holomorphic modular forms and modular stacks). We denote by $\widetilde{M}_*^{(\leq p)}, \widehat{M}_*^{(\leq p)}, \overrightarrow{M}_*^{(\leq p)}$ the subspaces of quasimodular forms (respectively almost holomorphic modular forms and modular fields) of depth $\leq p$ over a certain group Γ.

THEOREM 1. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete subgroup of finite covolume and p a positive integer. We have the isomorphisms :*

$$\begin{array}{lclcl} \widetilde{M}_*^{(\leq p)} & \simeq & \overrightarrow{M}_*^{(\leq p)} & \simeq & \widehat{M}_*^{(\leq p)} \\ f & \mapsto & (f_0, \cdots, f_p) & \mapsto & \sum_{j=0}^{p} \frac{f_j(z)}{(z-\overline{z})^j}, \end{array}$$

where the sequence of coefficients (f_j) are associated to f according to (2)*. The inverse map is given by $(f_0, \cdots, f_p) \longrightarrow f_0$.*

Remark 2. This theorem implies that an almost holomorphic modular form and a modular stack are determined by their first coefficient f_0, or first coordinate f_0 respectively.

3. The additive and $\mathrm{sl}_2(\mathbb{C})$-modules structure of rings of quasimodular forms

There exists three derivation operators on the spaces of quasimodular forms. By the isomorphisms of Theorem 1, we get the corresponding operators on the other spaces. We check that there exists a representation of the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ on the spaces $\widetilde{M}_*$, $\widehat{M}_*$ and $\overrightarrow{M}_*$ of quasimodular forms, almost holomorphic modular forms and modular stacks.

Proposition 1. *The operator D of derivation, with respect to z acts on the space of quasimodular forms. This operator increases the weight by 2 and the depth by 1. For any $k \geq 0$ and $p \geq 0$ we have:*

$$D : \widetilde{M}_k^{(\leq p)} \longrightarrow \widetilde{M}_{k+2}^{(\leq p+1)}.$$

Proof. Let $f \in \widetilde{M}_k^{(\leq p)}$. By definition we have:

$$(c\,z+d)^{-k} f(\frac{a\,z+b}{c\,z+d}) = \sum_{0\leq j\leq p} f_j(z)\ (\frac{c}{c\,z+d})^j,$$

with holomorphic maps f_j. So:

$$\begin{aligned} & (c\,z+d)^{-k-2}\ f'(\tfrac{a\,z+b}{c\,z+d}) \\ = \quad & D[(c\,z+d)^{-k}\ f(\tfrac{a\,z+b}{c\,z+d})] + kc\,(c\,z+d)^{-k-1}\ f(\tfrac{a\,z+b}{c\,z+d}) \\ = \quad & D[\textstyle\sum_{0\leq j\leq p} f_j(z)\ (\tfrac{c}{c\,z+d})^j] + \tfrac{kc}{c\,z+d}\textstyle\sum_{0\leq j\leq p} f_j(z)\ (\tfrac{c}{c\,z+d})^j \\ = \quad & \textstyle\sum_{0\leq j\leq p+1}[f_j'(z) + (k-j+1)f_{j-1}(z)]\ (\tfrac{c}{c\,z+d})^j. \end{aligned}$$

(with $f_{-1} \equiv f_{p+1} \equiv 0$). So the weight of f' is $k+2$ and its depth is $\leq (p+1)$. $\square$

Proposition 2. *If $f \in \widetilde{M}_k^{(\leq p)}$ is a quasimodular form and $F(z) = f_0(z) + \frac{f_1(z)}{z-\overline{z}} + \cdots + \frac{f_p(z)}{(z-\overline{z})^p}$ with $f_0 = f$ is the almost holomorphic modular form which corresponds to f, then any f_l is a quasimodular form of weight $k-2l$ and depth $\leq p-l$. In particular, we have a map $\delta : \widetilde{M}_k \longrightarrow \widetilde{M}_{k-2}$ which maps $\widetilde{M}_k^{(\leq p)}$ to $\widetilde{M}_{k-2}^{(\leq p-1)}$ for any p. This map is given by $f = f_0 \mapsto f_1$ and it has the following properties:*

(i) The kernel of the map $\delta : \widetilde{M}_k \longrightarrow \widetilde{M}_{k-2}$ is the space M_k.

(ii) If $f(z)$ is a quasimodular form, then the almost holomorphic modular form F associated to f is given by: $F(z) = \sum_{n=0}^{\infty} \frac{(\delta^n f)(z)}{n!\ (z-\overline{z})^n}$.

Remark 3. The sum in (ii) is finite because if the depth of f is $\leq p$ then $\delta^n(f) = 0$ for $n > p$. In fact, since $\widetilde{M}_k^{(\leq 0)} = M_k$ vanishes for $k < 0$, we see that the depth of a quasimodular form f of weight k is at most equal to $\frac{k}{2}$.

Proof. The second part (ii) is clear by using Definition 3 and Theorem 1. The first part (i) is a consequence of this since : $\delta(f) = 0 \Leftrightarrow \delta^n(f) = 0\ (\forall n \geq 1) \Leftrightarrow f = F \Leftrightarrow F$ holomorphic. $\square$

COROLLARY. *Let $k \geq 0$, $f \in \widetilde{M}_k^{(\leq p)}$ and $F = f_0 + \frac{f_1}{z-\overline{z}} + \cdots + \frac{f_p}{(z-\overline{z})^p}$ the associated almost holomorphic modular form. Then, we have $f_p \in M_{k-2p}$ and more generally $f_j \in \widetilde{M}_{k-2\,j}^{(\leq p-j)}$.*

Proof. By properties of δ, it is clear that $f_j \in \widetilde{M}_{k-2\,j}^{(\leq p-j)}$. In particular $f_p \in \widetilde{M}_{k-2\,p}^{(\leq 0)}$. Since a quasimodular form of depth 0 is modular, we deduce that $f_p \in M_{k-2\,p}$. $\square$

DEFINITION 5. Let $H : \widetilde{M}_* \longrightarrow \widetilde{M}_*$ be the operator which associates kf to any quasimodular form f of weight k, i.e. $H(f) = k\, f$.

PROPOSITION 3. *The operators D, δ and H satisfy the relations :*

$$\begin{array}{rc} i) & [H, D] = 2\, D. \\ ii) & [H, \delta] = -2\, \delta. \\ iii) & [\delta, D] = H. \end{array}$$

In other words, we have a representation of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ *over the spaces $\widetilde{M}_*$, $\widehat{M}_*$ and $\overrightarrow{M}_*$.*

Proof. The points (i) and (ii) follow from the fact applying D or δ, increases or decreases the weight by 2.

To prove (iii), we compute the braket $[\delta, D]$ over spaces of modular stacks. By using Theorem 1, we obtain the corresponding result over spaces of quasimodular forms. It is easy to check that, for a modular stack of weight k and depth $\leq p$, we have the property:

$$\delta(f_0, \cdots, f_p) = (f_1, 2f_2, \cdots, jf_j, \cdots).$$

We deduce from Proposition 2 that:

$$\begin{aligned} D\delta(f_0, \cdots, f_j, \cdots, f_p) &= D(f_1, 2f_2, \cdots, pf_p) \\ &= (f_1', \cdots, (j+1)f_{j+1}' + (k-1-j)jf_j, \cdots). \end{aligned}$$

On the other hand,

$$D(f_0, \cdots, f_j, \cdots, f_p) = (f_0', f_1' + k\, f_0, \cdots, f_j' + (k-j+1)f_{j-1}, \cdots).$$

So,

$$\delta D(f_0,\cdots,f_p)=(f_1'+k\ f_0,\cdots,(j+1)f_{j+1}'+(j+1)(k-j)f_j,\cdots).$$

By taking the difference of the two preceding equations (giving δD and $D\delta$ for a modular stack), we find:

$$[\delta,D](f_0,\cdots,f_p)=(k\ f_0,\cdots,kf_j,\cdots)=k(f_0,\cdots,f_p),$$

which implies (by isomorphisms of Theorem 1) the property:

$$[\delta,D](f)=H(f).$$

□

Let $\mathcal{U}$ be the universal envelopping algebra of $\mathrm{sl}_2(\mathbb{C})$. We compute next the class of the operator $\delta^n D^n$ modulo $\mathcal{U}\delta$, for any $n\in\mathbb{N}$.

Proposition 4. *The class of the operator $\delta^n D^n$ modulo $\mathcal{U}\delta$ is given by :*

$$(\delta^n\ D^n)\equiv n!\prod_{j=0}^{n-1}(H+j)(mod\,\mathcal{U}\delta)\qquad .$$

Proof. By Proposition 3, we know that $\delta D\equiv H(\mathrm{mod}\ \mathcal{U}\delta)$. Let $j>1$, and suppose that $\delta^{j-1}\ D^{j-1}\equiv P_{j-1}(H)(\mathrm{mod}\ \mathcal{U}\delta)$ where P_{j-1} is a polynomial of degree $(j-1)$. We have:

$$\begin{aligned}\delta^jD^j &\equiv \delta^{j-1}(\delta D)\ D^{j-1}\equiv\delta^{j-1}(D\delta+H)D^{j-1}\\ &\equiv \delta^{j-2}(\delta D)\delta D^{j-1}+\delta^{j-1}HD^{j-1}\\ &\equiv \delta^{j-2}\ D\delta^2D^{j-1}+\delta^{j-2}H\delta D^{j-1}+\delta^{j-1}HD^{j-1}\\ &=\cdots\\ &\equiv \delta D\ \delta^{j-1}\ D^{j-1}+\textstyle\sum_{n=1}^{j-1}\delta^n\ H\ \delta^{j-1-n}\ D^{j-1}(\mathrm{mod}\,\mathcal{U}\delta).\end{aligned}$$

In the other hand,

$$\begin{aligned}\delta^nH &\equiv \delta^{n-1}H\delta+2\delta^n\quad\text{(By Proposition 4)}\\ &\equiv \delta^{n-2}\ H\ \delta^2+2(2\delta^n)\equiv\cdots\\ &\equiv n(2\delta^n)+H\delta^n(\mathrm{mod}\,\mathcal{U}\delta).\end{aligned}$$

So we have:

$$\begin{aligned}\delta^j\ D^j &\equiv \delta D\ \delta^{j-1}\ D^{j-1}+\textstyle\sum_{n=1}^{j-1}(H+2n)\delta^{j-1}D^{j-1}\\ &\equiv \delta D\ \delta^{j-1}\ D^{j-1}+(j-1)(H+j)\delta^{j-1}D^{j-1}(\mathrm{mod}\,\mathcal{U}\delta),\end{aligned}$$

Finally:

$$\begin{aligned}P_j(H) &\equiv HP_{j-1}(H)+(j-1)(H+j)P_{j-1}(H)\\ &\equiv j(H+(j-1))\ P_{j-1}(H)(\mathrm{mod}\,\mathcal{U}\delta).\end{aligned}$$

We obtain the result: $P_n\equiv n!\prod_{j=0}^{n-1}(H+j)(\mathrm{mod}\,\mathcal{U}\delta)$ by induction. □

COROLLARY. *Let $f \in M_k$ a modular form of weight k and $n \geq 0$. We have an exact sequence:*

$$\delta^n\, D^n(f) = n!^2 \binom{k+n-1}{n} f.$$

Proof. By using (i) of Proposition 2 , $f \in \ker(\delta)$. So the last Proposition implies the result. □ □

PROPOSITION 5. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and finite covolume subgroup. Let $k \geq 0$ and $p \geq 0$ be integers, if $p < \frac{k}{2}$ then:*

$$\widetilde{M}_k^{(\leq p)} = D^p(M_{k-2p}) \oplus \widetilde{M}_k^{(\leq p-1)}.$$

Proof. By Proposition 2 and its corollary we have, $\delta^p(f) \in M_{k-2p}$. By application of the corollary of Proposition 4 to $\delta^p(f)$ we get:

$$p!^2 \binom{k-p-1}{p} f - D^p(\delta^p(f)) \in \widetilde{M}_k^{(\leq p-1)}(\Gamma).$$

In particular, if $k > 2p$ then f is the sum of the *pth* derivative of a modular form and of a quasimodular form of depth $\leq p-1$. □

We finish this section by giving an additive structure theorem and an $\mathrm{sl}_2(\mathbb{C})$-module structure theorem for rings of quasimodular forms over discrete and finite covolume subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

THEOREM 2. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and finite covolume subgroup. We have an exact sequence:*

$$0 \longrightarrow M_2(\Gamma) \longrightarrow \widetilde{M}_2(\Gamma) \xrightarrow{\delta} \mathbb{C}.$$

Then we have:

$$\widetilde{M}_* = \mathbb{C} \oplus \bigoplus_{i=0}^{\infty} (D^i(M_*)) \oplus \begin{cases} 0 & \textit{if } \dim(\mathrm{Im}(\delta)) = 0. \\ \oplus_{i=0}^{\infty} \mathbb{C} D^i(\phi) & \textit{if there exists } \phi \in \widetilde{M}_2(\Gamma) \\ & \textit{satisfying } \delta(\phi) = 1. \end{cases}$$

Proof. By general properties of quasimodular forms, it is clear that $M_2(\Gamma) \subset \widetilde{M}_2(\Gamma)$ and $\delta(M_2(\Gamma)) = 0$.

We suppose that $\dim \mathrm{Im}(\delta) = 0$. Then for any $f \in \widetilde{M}_k^{\leq p}$, we have $p < \frac{k}{2}$. We deduce from Proposition 5 that f is the direct sum of the pth derivative of a modular form and of a quasimodular form of depth $< p$. So by induction on p, we get that $\widetilde{M}_k = \bigoplus_{i=0}^{\frac{k}{2}} D^i M_{k-2i}$. This implies the theorem in the case $\dim(\mathrm{Im}(\delta)) = 0$.

We suppose know that there exists ϕ such that $\delta(\phi) = 1$. By using the same proof for the additive structure of $\widetilde{M}_*(\Gamma_1)$ (as in [4]), we deduce the theorem in the second case. □

For $k>0$, let $\mathcal{U}_k$ be the $\mathrm{sl}_2(\mathbb{C})$-module defined by a basis $(x_j^{(k)})_{j\in\mathbb{N}}$ with $Dx_j^{(k)} = x_{j+1}^{(k)}$, $Hx_j^{(k)} = (k+2j)x_j^{(k)}$ and

$$\delta x_j^{(k)} = \begin{cases} j(k+j-1)x_{j-1}^{(k)} & \text{if } j \geq 1. \\ 0 & \text{if } j = 0. \end{cases}$$

We define $\mathcal{U}_0 = \mathbb{C}$ with the trivial action of $\mathrm{sl}_2(\mathbb{C})$. Finally, if there exists ϕ such that $\delta(\phi) = 1$, we define an extension $\widehat{U}_2$ of $\mathcal{U}_2$ (i.e. $\widehat{U}_2 \simeq \mathbb{C}\oplus\mathcal{U}_2$). If $(\hat{x}_j)_{j\in\mathbb{N}}$ is a basis of $\widehat{U}_2$ then the the action of $\mathrm{sl}_2(\mathbb{C})$ is defined as over a basis of $\mathcal{U}_2$ except that $\delta\hat{x}_0 = 1$.

For any $k>0$, we have an embedding:

$$\begin{array}{ccc} M_k\otimes\mathcal{U}_k & \longrightarrow & \widetilde{M}_* \\ f\otimes x_j^{(k)} & \longrightarrow & D^j f. \end{array}$$

In the case $k=0$, we have a map $\mathbb{C}\otimes\mathbb{C}\longrightarrow\mathbb{C}$. Finally

$$\begin{array}{ccc} \mathbb{C}\phi\otimes\widehat{U}_2 & \longrightarrow & \widetilde{M}_* \\ \phi\otimes\hat{x}_i & \longrightarrow & D^i\phi \text{ if } i\geq 1 \\ \phi\otimes 1 & \longrightarrow & 1 \text{ if } i=0 \end{array}$$

Theorem 3. *Let $\Gamma\subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and finite covolume subgroup. Then we have:*

$$\widetilde{M}_*(\Gamma) = \bigoplus_{k=0}^{\infty} M_k(\Gamma)\otimes\mathcal{U}_k \oplus \begin{cases} 0 & \textit{if } \delta(\widetilde{M}_2(\Gamma)) = 0. \\ \widehat{U}_2 & \textit{if there exists } \phi\in\widetilde{M}_2(\Gamma) \\ & \textit{such that } \delta(\phi) = 1. \end{cases}$$

The proof of this theorem use the definitions of the maps $M_k\otimes\mathcal{U}_k \longrightarrow \widetilde{M}_*$, and Theorem 2.

4. The cocompact / non-cocompact dichotomy

We prove a dichotomy Theorem which characterizes cocompact groups in terms of their space of quasimodular forms of weight 2. This fundamental dichotomy implies the condition $p<\frac{k}{2}$ in the case of cocompact groups and we deduce from it the additive structure theorem in the cocompact case.

Theorem 4. **Let $\Gamma\subset\mathrm{PSL}(2,\mathbb{R})$ be a discrete and finite covolume subgroup. If Γ is not cocompact, there exists a quasimodular form ϕ of weight 2 over Γ; morover ϕ is not modular and $\widetilde{M}_2 = M_2(\Gamma)\oplus\mathbb{C}\phi$. If Γ is cocompact we have $\widetilde{M}_2(\Gamma) = M_2(\Gamma)$.**

Proof. In the case of a group Γ which is commensurable with $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, we take ϕ equal to the restriction of the Eisenstein series E_2 if Γ is a subgroup of congruence of Γ_1 and in the other cases of a group

Γ commensurable with Γ_1, take a normalised trace of E_2 and get a quasimodular form of weight 2 over Γ with $\delta(\phi) = 1$. In particular, ϕ is not modular. If Γ is not an arithmetic group (such as a Hecke modular group) then we can define ϕ as the quasimodular form associated to an almost holomorphic modular form $E_{2,\Gamma}(z, 0)$ of weight 2 defined as the limit in s of a family $E_{2,\Gamma}(z, s)$ of almost holomorphic modular forms of weight 2.

We suppose that Γ is cocompact and that there exists a quasimodular form f of weight 2 which is not modular. Let F be the almost holomorphic modular form associated to f. We have :

$$F(z) = f(z) + \frac{c}{z - \overline{z}} \text{ with } c \neq 0,$$

in fact, $f_0 = f \in M_2$ so $f_1 \in M_0 = \mathbb{C}$. Let $\omega(z) = F(z)\ dz$, the modularity of F implies the Γ invariance of the ω form. So this $1-$form is defined on the quotient $X = \mathcal{H}/\Gamma$. On the other hand, we have

$$d\omega = -\frac{\partial F}{\partial \overline{z}}\ dz \wedge d\overline{z} = -\frac{c}{(z - \overline{z})^2}\ dz \wedge d\overline{z}.$$

This means that $d\omega$ is a multiple of the volume form. So there exists $\alpha \neq 0$ such that :

$$0 \neq \alpha \ \mathrm{Vol(X)} = \int_{\mathrm{X}} \mathrm{d}\omega.$$

On the other hand $\int_X d\omega = 0$, this equality is a consequence of Stokes's formula and the fact that X is a variety without boundary. We obtain a contradiction. $\square$

Remark 4. Theorem 4 implies that $\delta(\widetilde{M_2}) = \delta(M_2) = 0$ in the cocompact case. By Theorem 2 we deduce the corresponding additive structure for rings of quasi-modular forms. In the non-cocompact case $\delta(\widetilde{M_2}) = \mathbb{C}$ and by Theorem 2, we deduce from this the corresponding additive structure.

5. Rings of quasimodular forms

For a discrete and cocompact subgroup Γ we denote by I (respectively $\widetilde{I}$) the ideal of modular forms (respectively quasimodular forms) over Γ of positive weights. Finally, $\widetilde{I}^2_k = \sum\limits_{0<j<k} \widetilde{M}_j \widetilde{M}_{k-j}$ is the $\mathbb{C}$-vector space of decomposable quasimodular forms of weight k.

Theorem 5. **Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cocompact subgroup. Let $\epsilon = \dim_{\mathbb{C}} I/(I \cap \widetilde{I}^2)$ and let $\{A_1, \cdots, A_\epsilon\}$ be homogeneous elements of I of weights $w_1, \cdots, w_\epsilon \in 2\mathbb{Z}$, which are**

linearly independent modulo $(\widetilde{I})^2$**. Then for any** $k \geq 0$,

$$(\widetilde{I}/\widetilde{I}^2)_k = \bigoplus_{i=1, w_i \leq k}^{\epsilon} \mathbb{C}D^{(\frac{k-w_i}{2})}(A_i) \qquad .$$

Proof. We denote by P_s, $(s = 2, 4, \cdots)$ the vector space generated by all the A_i of weight $w_i = s$ and we write $\delta_i = \dim P_i$ so $\sum_i \delta_i = \epsilon$. We have the commutative diagram:

$$\begin{array}{ccc} P_s & \hookrightarrow & M_s \\ & \searrow & \downarrow \mathrm{D}^n \\ & & \widetilde{M}_{s+2n} \end{array}$$

The maps which appear in the last diagram are injective. In fact $P_s \subset M_s$. On the other hand $D^n(f) = 0$ implies that f is a polynomial. This polynomial is zero because it defines a modular form f of positive weight. We deduce:

$$\dim D^n(P_s) = \delta_s \quad .$$

On the other hand $D^{\frac{k-2}{2}}(P_2) \subset D^{\frac{k-2}{2}}(M_2), \cdots, D^{\frac{k-w_\epsilon}{2}}(P_{w_\epsilon}) \subset D^{\frac{k-w_\epsilon}{2}}(M_{w_\epsilon})$. Since Theorem 2 implies that the sum of spaces $D^{\frac{k-s}{2}}M_s$, for $s = 2, 4, \cdots, w_\epsilon$, is a direct sum, we get that the sum of their subspaces $D^{\frac{k-s}{4}}P_s$ is also direct. On the other hand, for any $n \geq 0$ and s : $2 \leq s \leq w_\epsilon$ we have,

$$D^n(P_s) \cap (\widetilde{I})^2 = 0 \qquad .$$

Indeeed, by corollary of Proposition 4,

$$\text{for all } f \in P_s \text{ , } \delta^n D^n(f) = c_n\, f \text{ with } c_n \neq 0 \quad .$$

Since δ is a derivation, we have:

$$\text{for all } f, g \in \widetilde{I}, \delta(gh) = \delta(g)h + g\delta(h) \qquad .$$

This implies $\delta(\widetilde{I}^2) \subset \widetilde{I}^2$, indeed $M_0 \cap Im(\delta) = 0$ because $\widetilde{M_2} = M_2$. It remains to prove that : for any $f_2 \in P_2, \cdots, f_{w_\epsilon} \in P_{w_\epsilon}$, if $f_2^{(\frac{k-2}{2})} + \cdots + f_{w_\epsilon}^{(\frac{k-w_\epsilon}{2})} \in \widetilde{I}^2$ then $f_2 = \cdots = f_{w_\epsilon} = 0$. We write $\alpha_2 = \frac{k-2}{2}, \cdots, \alpha_{w_\epsilon} = \frac{k-w_\epsilon}{2}$ and we suppose that:

$$f_2^{(\alpha_2)} + \cdots + f_{w_\epsilon}^{(\alpha_\epsilon)} \in \widetilde{I}^2,$$

we can suppose also that:

$$\alpha_2 \geq \alpha_4 \geq \cdots \geq \alpha_{w_\epsilon}.$$

We apply the operator δ^{α_2}, then all $f_i^{(\alpha_i)}$ with $i > 2$ vanish. We deduce from this that $f_2 \in \delta^{(\alpha_2)}(\widetilde{I}^2) \subset \widetilde{I}^2$ and so $f_2 = 0$. We restart with the

operator δ^{α_4}, we prove that $f_4 = 0$ and by induction, we deduce that $f_{w_\epsilon} = 0$. $\square$

Corollary. **Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cocompact subgroup. Let $\epsilon = \dim_{\mathbb{C}} I/(I \cap \widetilde{I}^2)$ and let $\{A_1, \cdots, A_\epsilon\}$ be homogeneous elements of I, linearly independent modulo $\widetilde{I}^2$ of weights $w_1, \cdots, w_\epsilon$ respectively, then:**

$$\dim_{\mathbb{C}}(\widetilde{I}/\widetilde{I}^2)_k = \epsilon, \text{ for all } k \geq \max_i\{w_i\}$$

In particular, $\widetilde{M_*}$ is a non-finitely generated $\mathbb{C}$-algebra.

Remark 5. This result is false in the non-cocompact case. For example for $\mathrm{PSL}(2,\mathbb{Z})$, we have $\widetilde{M_*} \simeq \mathbb{C}[E_2, E_4, E_6]$ where E_2, E_4 and E_6 are the Eisenstein series of weights 2,4 and 6, respectively.

Proof. The fact that $\widetilde{M_*}$ is not finitely generated is equivalent to saying that $\dim((\widetilde{I}/\widetilde{I}^2)_k) = 0$, for k large enough. The corollary is a consequence of Theorem 3. $\square$

Remark 6. We finish this section by observing that the ideas used to prove our two theorems can be used to prove more general results about the additive and multiplicative structure of the differential closure $\mathrm{CL}(\mathcal{M})_*$ of any ring $\mathcal{M}_*$ different from $\mathbb{C}$ and generated by a finite set of holomorphic modular forms or meromorphic modular forms of positive weight. We will define the differential closure, and we give the corresponding results (Theorems 6 and 7). The main point for Theorem 7 is the fact that $\mathrm{CL}(\mathcal{M})_2 = \mathcal{M}_2$.

Definition 6. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete subgroup of finite covolume, and let $\mathcal{M}_*$ be a graded subring of the ring of meromorphic modular forms over Γ. The differential closure $\mathrm{CL}(\mathcal{M})_*$ of $\mathcal{M}_*$ is the smallest ring which contains $\mathcal{M}_*$ and is closed under the derivation D, with graduation $D^j\mathcal{M}_k \subset \mathrm{CL}(\mathcal{M})_{\mathrm{k+2j}}$.

Notation 2. We denote by $J_{\mathcal{M}}$ the ideal of elements in $\mathrm{CL}(\mathcal{M})_*$ of positive weight and by $I_{\mathcal{M}}$ the ideal of $\mathcal{M}_*$ of elements of positive weight. The ideal $J_{\mathcal{M}}^2$ is the ideal of decomposable forms in $\mathrm{CL}(\mathcal{M})_*$.

Theorem 6. **Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete subgroup of finite covolume, and let $\mathcal{M}_*$ be a ring generated by a finite set of holomorphic modular forms or meromorphic modular forms of positive weight over Γ. Then for any $k \geq 0$:**

$$\mathrm{CL}(\mathcal{M})_{\mathrm{k}} = \bigoplus_{0 \leq \mathrm{j} \leq \frac{\mathrm{k}}{2}} \mathrm{D}^{\mathrm{j}}\mathcal{M}_{\mathrm{k-2j}}.$$

THEOREM 7. **Let $\mathcal{M}_*$ be a ring like the one of the last theorem. Let $\epsilon = \dim_{\mathbb{C}} I_{\mathcal{M}}/(I_{\mathcal{M}} \cap J^2_{\mathcal{M}})$ and let $\{f_1, \cdots, f_\epsilon\}$ be homogeneous elements of $I_{\mathcal{M}}$, linearly independent modulo $J^2_{\mathcal{M}}$ of weights $l_1, \cdots, l_\epsilon$, respectively. Then for any k even:**

$$(J_{\mathcal{M}}/J^2_{\mathcal{M}})_k = \bigoplus_{\substack{i=1,\cdots,\epsilon \\ l_i \leq k}} \mathbb{C}D^{(\frac{k-l_i}{2})}(f_i).$$

In particular,

$$\dim_{\mathbb{C}}(J_{\mathcal{M}}/J^2_{\mathcal{M}})_k = \epsilon, \qquad \textbf{for all } k \geq \max\{l_1, \cdots, l_\epsilon\},$$

and the ring $\mathrm{CL}(\mathcal{M})_*$ is not finitely generated as a $\mathbb{C}$ algebra.

6. Algebraic characterization of cocompact modular groups

We recall that a Poisson algebra is a commutative and associative algebra A with a Lie structure, i.e. a bilinear operation $[\cdot, \cdot] : A \times A \longrightarrow A$ satisfaying the Jacobi identity, such that for any $x \in A$, the map $[x, \cdot]$ is a derivation. If furthermore $A = \bigoplus_{n\geq 0} A_n$ is graded with $A_m A_n \subset A_{m+n}$, $[A_m, A_n] \subset A_{m+n+1}$, then A is called a graded Poisson algebra.

Examples. 1) Let A be a graded algebra (commutative and associative) and let $d : A \longrightarrow A$ be a derivation of degree 1, i.e. $d(A_n) \subset A_{n+1}$ and $d(xy) = xd(y) + yd(x)$, for evry $x, y \in A$. Then the braket defined by $[x, y] = H(x)d(y) - H(y)d(x)$, where H is the operator of multiplication by the weight n in A_n satisfies the Jacobi identity (a simple verification) and has the property that: $x \longrightarrow [x, y]$ is a derivation for evry fixed $y \in A$ (because H and d are derivations). We call a Poisson algebra *trivialisable*, if it can be obtained in this way.

2) Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and finite covolume subgroup and let $A = M_{ev} = \bigoplus_{n\geq 0} M_{2n}$ (i.e., $A_n = M_{2n}$) be the graded algebra of modular forms. This algebra has a Poisson structure with the usual multiplication and where the braket $[\cdot, \cdot] = [\cdot, \cdot]_1$, is the first Rankin-Cohen braket.

THEOREM 8. **Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and finite covolume subgroup. Then the Poisson algebra $(M_{ev}(\Gamma), [\cdot, \cdot]_1)$ is trivialisable if and only if Γ is not cocompact.**

We use the next two lemmas, where the second is a corollary of the first.

LEMMA. *Let M_*^{mer} be the ring of meromorphic modular forms over a discrete and finite covolume group, then any derivation $\partial : M_*^{\mathrm{mer}} \longrightarrow M_{*+2}^{\mathrm{mer}}$ trivialising the first Rankin-Cohen braket has the form $\partial = D - \phi E$ where E is the Euler operator (multiplication by the weight) and $\phi \in \widetilde{M}_2^{\mathrm{mer}}$ with $\delta\phi = 1$.*

Proof. For any discrete finite covolume group Γ there exists a meromorphic quasimodular form ψ of weight 2 such that $\delta\psi = 1$: we can divide the logarithmic derivative of any non zero modular form by its weight. Then, we consider $\partial_\psi = D - \psi E$; this operator trivialises the first braket. We suppose also that ∂ trivialises the first Rankin-Cohen braket. Then for any $f \in M_k^{mer}$ and $g \in M_l^{mer}$, we have the relation:

$$\frac{\partial f - \partial_\psi f}{E(f)} = \frac{\partial g - \partial_\psi g}{E(g)} \qquad .$$

In particular, there exists a holomorphic modular form α of weight 2 such that $\partial - \partial_\psi = \alpha E$. This implies $\partial = D - (\alpha + \psi)E$. Since $\psi + \alpha$ is a meromorphic quasimodular form of weight 2 and $\delta(\psi + \alpha) = \delta\psi = 1$ (because α is modular), we can take $\phi = \psi + \alpha$. $\square$

LEMMA. *Let M_* be the ring of modular forms over a discrete and finite covolume group Γ. Then any derivation $\partial : M_* \longrightarrow M_{*+2}$ trivialising the first Rankin-Cohen braket has the form $\partial_\phi = D - \phi E$ with $\phi \in \widetilde{M}_2$ and $\delta\phi = 1$.*

Proof. By the last lemma, ∂ has the form ∂_ϕ with $\phi \in M_2^{\mathrm{mer}}(\Gamma)$. The fact that $\partial_\phi(f)$ must be holomorphic for any holomorphic modular form f implies that ϕ must also be holomorphic, since different modular forms over Γ cannot have the same set of zeros. $\square$

The proof of Theorem 6 is a consequence of corollary of Proposition 3 and of the last lemma, because holomorphic quasimodular forms of weight 2 over a cocompact modular group are modular (see Theorem4).

Remark 7. This theorem is false in the non-cocompact case. for example, if $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ is the classical modular group; it then exists a derivation $\partial = D - \frac{E_2}{12}E$, where E_2 is the Eisenstein series of weight 2, which trivialise the first Rankin-Cohen braket.

7. EMMBEDDING OF QUASIMODULAR FORMS IN FINITELY GENERATED RINGS

THEOREM 9. **Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cocompact subgroup. Then there exists a quasimodular form ϕ of weight 2 over Γ satisfying $\delta(\phi) = 1$, with simple poles in the orbit**

of i, and without other poles. For any such form ϕ we have $\mathrm{Res}_{z=i}(\phi(z)dz) = \mathcal{K}$ for any α in the Γ-orbit of i with $\mathcal{K} = \frac{\mathrm{Vol}(\mathcal{H}/\Gamma)}{4\pi}$.

Remark 8. The form ϕ is unique up to the addition of an holomorphic modular form of weight 2 (the dimension of the space of such forms is equal to the genus g of the Riemann surface $\mathcal{H}/\Gamma$).

After conjugating Γ in $\mathrm{PSL}(2,\mathbb{R})$, we can replace $''i''$ in the theorem by any other point $z_0 \in \mathcal{H}$.

Proof. First, we suppose that Γ acts on $\mathcal{H}$ without fixed points (this means that the action is free). Let f be a non zero modular form of weight $k > 0$, we know that $\frac{f'}{f}$ is a meromorphic quasimodular form of weight 2 with $\delta(\frac{f'}{f}) = k \neq 0$. Moreover the poles of $\frac{f'}{f}$ are simple and Γ invariant. We denote by $\{P_1, \cdots, P_n\}$ the poles of $\frac{f'}{f}$ in $\mathcal{H}/\Gamma$ different from i. We want to construct a meromorphic modular form h of weight 2 such that the sum $\frac{f'}{f} + h$ has no poles outside the orbit of i. Let $X = \mathcal{H}/\Gamma$ the Riemann compact surface (of genus g). Then the hypothesis on Γ implies that X is smooth and that $g > 1$. We denote by Ω^1_X the sheaf of holomorphic differential 1-forms over X. For any set of distinct points $\{q_1, \cdots, q_m\} \subset X$ (with $m \geq 1$), we denote by $\Omega^1_X(q_1 + \cdots + q_m)$ the sheaf of holomorphic differential $1-$ forms over X with simple poles at $q_1, \cdots, q_m$. We will prove:

$$H^0(X, \Omega^1_X(q_1 + \cdots + q_m)) \simeq \mathbb{C}^{g+m-1}.$$

Let K be the canonical divisor of X. By the Riemann-Roch Theorem we have:

$$l(K+q_1+\cdots+q_m) = l(-(q_1+\cdots+q_m)) + \deg(K+q_1+\cdots+q_m) - g + 1.$$

From $\deg(K) = 2g-2$ and $l(-(q_1+\cdots+q_m)) = 0$, we deduce that :

$$l(K + q_1 + \cdots + q_m) = g + m - 1 \quad .$$

If we apply the Riemann Roch Theorem to the cases $m = 1$ and $m = n+1$, we obtain the exact sequence:

$$0 \longrightarrow H^0(X, \Omega^1_X(i)) \longrightarrow H^0(X, \Omega^1_X(i + P_1 + \cdots + P_n)) \xrightarrow{\mathrm{Res}} \mathbb{C}^n \longrightarrow 0,$$

where Res maps a differential 1-form ω to $(\mathrm{Res}_{\mathrm{P_1}}(\omega), \cdots, \mathrm{Res}_{\mathrm{P_n}}(\omega))$. So we can choose h of weight 2 such that $\phi = \frac{1}{k}\frac{f'}{f} + h$ has a simple pole at i and no poles outside the orbit of i. We also have $\delta\phi = 1$.

To compute the constant $\mathcal{K}$ we apply Stokes's formula to the meromorphic differential $1-$form $\omega(t) = \phi^*(t+i)\, dt$ over X where $\phi^*(t+i) = \phi(t+i) + \frac{1}{t-\bar{t}}$ is the almost holomorphic modular form associated to ϕ.

Let U_ϵ be a disk with center i and radius ϵ included in X. By Stokes's formula, we have:

$$\int_{X-U_\epsilon} d\omega(t) = \int_{\partial(X-U_\epsilon)} \omega(t)$$

since, $d\omega(t) = d\phi^*(t) \wedge dt = -\frac{\partial\phi^*(t+i)}{\partial\bar{t}} dt \wedge d\bar{t}$. On the other hand ϕ is holomorphic over $\mathcal{H}$ so:

$$-\frac{\partial\phi^*}{\partial\bar{t}} = \frac{\partial}{\partial\bar{t}}(\frac{1}{t-\bar{t}})\, dt \wedge d\bar{t}.$$

because ϕ satisfies $\frac{\partial}{\partial\bar{t}}\phi = 0$. We obtain $d\omega(t) = \frac{dt\wedge d\bar{t}}{(t-\bar{t})^2}$, or $\frac{1}{2i}$ times the volume form, so $\int_{X-U_\epsilon} d\omega(t) = \frac{1}{2i}\mathrm{Vol(X-U_\epsilon)}$. On the other hand $\int_{\partial(X-U_\epsilon)} \omega(t) = -\int_{\partial U_\epsilon} \omega(t)$ because X is a compact manifold without boundary. So $\int_{\partial(X-U_\epsilon)} \omega(t) = -\int_{\partial(U_\epsilon)} \phi(t+i) + O(1)$, indeed $\phi^*(t+i) - \phi(t+i)$ is a continuous map over ∂U_ϵ. In the other hand $\phi(t+i) \sim \frac{\mathcal{K}}{t}$ so $-\int_{\partial U_\epsilon} \omega(t) = -(2\pi i)\mathcal{K} + O(1)$; by letting ϵ to 0 we obtain $\mathcal{K} = \frac{\mathrm{Vol(X)}}{4\pi}$. This completes the proof in the case of groups acting on $\mathcal{H}$ without fixed points.

We suppose now that Γ acts on $\mathcal{H}$ and that the action is not necessarily free. Selberg lemma implies that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index without torsion. The first part of the proof implies that there exists a quasimodular form α over Γ' of weight 2 with at most simple poles in the orbit of the point i. We define:

$$\beta(z) = \sum_{\gamma\in\Gamma/\Gamma'} [(\alpha \mid \gamma)(z) - \frac{c}{cz+d}],$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $(\alpha \mid \gamma)(z) = (cz+d)^{-2}\alpha(\frac{az+b}{cz+d})$. We will prove that β is a quasimodular form over Γ of weight 2. Let α^* be the almost holomorphic modular form associated to α. It is easy to check that

$$\beta^*(z) = \sum_{\gamma\in\Gamma/\Gamma'} (\alpha^* \mid \gamma)(z),$$

is an almost holomorphic modular form over Γ of weight 2 (since α^* is modular, β^* corresponds to the trace of α^* over the group Γ). On the other hand we have $(\alpha^* \mid \gamma)(z) = [(\alpha \mid \gamma)(z) - \frac{c}{cz+d}] + \frac{1}{z-\bar{z}}$. So:

$$\beta^*(z) = \sum_{\gamma\in\Gamma/\Gamma'} [(\alpha \mid \gamma)(z) - \frac{c}{cz+d}] + \sum_{\gamma\in\Gamma/\Gamma'} \frac{1}{z-\bar{z}},$$

in other words $\beta^*(z) = \beta(z) + \frac{[\Gamma:\Gamma']}{z-\bar{z}}$. This proves that β is a quasimodular form over Γ of weight 2 and $\delta(\beta) = [\Gamma : \Gamma']$. It is clear that β has at

most simple poles on the orbite of i. Hence, $\frac{\beta}{[\Gamma:\Gamma']}$ is an appropriate form over Γ. $\square$

Notation 3. We denote by $M_*(\Gamma;\{i\})$ the ring of modular forms without poles outside the orbit of i and we denote by $M_*^{(\geq\alpha)}(\Gamma;i)$ the subset of modular forms over Γ with vanishing order at least equal to α at i. Finally we denote by $\widetilde{M}_2(\Gamma;\{i\})$, the space of quasimodular forms of weight 2 over Γ with all poles in the orbit of i.

LEMMA. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete cocompact subgroup, and ϕ a quasimodular form over Γ with at most simple poles in the orbit of i, and $\delta(\phi) = 1$. Then, we have $\mathrm{Res}_i(\phi(z)dz) = \frac{\mathrm{Vol}(\mathcal{H}/\Gamma)}{4\pi}$ and $\omega = \phi' - \phi^2$ is a modular form of weight 4 with at most double poles in the orbit of i.*

Proof. We know that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have:

$$\phi(\frac{az+b}{cz+d}) = (cz+d)^2\phi(z) + c\,(cz+d) \qquad .$$

By derivation, we get:

$$\phi'(\frac{az+b}{cz+d}) = (cz+d)^4\phi'(z) + 2c(cz+d)^3\phi(z) + c^2(cz+d)^2.$$

On the other hand:

$$\phi^2(\frac{az+b}{cz+d}) = (cz+d)^4\phi^2(z) + 2c(cz+d)^3\phi(z) + c^2(cz+d)^2,$$

this implies:

$$(\phi' - \phi^2)(\frac{az+b}{cz+d}) = (cz+d)^4(\phi' - \phi^2)(z).$$

So ω is a modular form of weight 4. Since $\phi'(i+x) \sim -\mathcal{K}\ x^{-2}$ and $\phi^2(i+x) \sim \mathcal{K}^2x^{-2}$ (for $x \to 0$), we deduce that:

$$\omega(x+i) \sim -\mathcal{K}(\mathcal{K}+1)x^2.$$

$\square$

PROPOSITION 6. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete cocompact subgroup, and ϕ a quasimodular form of weight 2 over Γ with $\delta(\phi) = 1$, which is holomorphic outside the orbit of i. Then there exists an operator*

$$D_\phi : M_k(\Gamma;\{i\}) \longrightarrow M_{k+2}(\Gamma;\{i\}),$$

defined by $D_\phi(f) = f' - k\phi f$. If ϕ has a simple pole at i, then $\mathrm{ord_i(D_\phi(f))} \geq \mathrm{ord_i(f)} - 1$, with inequality if and only if $\mathrm{ord_i(f)} = \mathrm{k}\,\mathcal{K}$, where k is the weight of f.

Remark 9. The case $\mathrm{ord_i(D_\phi(f))} = \infty$, can only happen if $\mathrm{ord_i(f)} = \mathrm{k(f)}\mathcal{K}$ where $k(f)$ is the weight of f.

Proof. Let f be a meromorphic modular form over Γ of weight k, then for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$. By derivation, we obtain:

$$Df(\frac{az+b}{cz+d}) = (cz+d)^{k+2} Df(z) + kc(cz+d)^{k+1} f(z).$$

On the other hand:

$$(\phi.f)(\frac{az+b}{cz+d}) = (cz+d)^{k+2}(\phi.f)(z) + c(cz+d)^{k+1}\ f(z).$$

This implies $D_\phi(f)(\frac{az+b}{cz+d}) = (cz+d)^{k+2} D_\phi(f)(z)$. On other words $D_\phi(f)$ is a modular form of weight $k+2$. In the other hand, if $f(x) \sim x^\alpha$ (with $\alpha \neq \mathcal{K}k$) then $D_\phi(f)(x) \sim (\alpha - k\ \mathcal{K})\ x^{\alpha-1}$. $\square$

We recall that I is the ideal of modular forms of positive weight over the group Γ.

Theorem 10. **Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete cocompact subgroup, let $\phi \in \widetilde{M_2}(\Gamma; \{i\})$ with $\delta(\phi) = 1$ and $\omega = \phi' - \phi^2$. Let $(f_1, \cdots, f_d)$ be a basis of I/I^2. Then there exists $N \in \mathbb{N}^*$ such that the ring R generated by the set:**

$$\{D_\phi^j(f_i)\ (1 \leq j \leq N, 1 \leq i \leq d)\ ; D_\phi^l(\omega)\ (1 \leq l \leq N)\},$$

is closed under the operator D_ϕ.

To prove the theorem, we use a lemma which describe finitely generated semigroups of $\mathbb{R}^2$:

Lemma. *Let G be a finitely generated semigroup of $\mathbb{R}^2$. We suppose that the group generated by G is a lattice $\Lambda \subset \mathbb{R}^2$ of rank 2. Let S be the sector $< G \cdot \mathbb{R}_+ >$. We suppose that S is convex, with angle at most equal to π. Then there exists $A \in S$ such that $(A + S) \cap \Lambda \subset G$.*

Proof. Let $\{P_1, \cdots, P_m\}$ be a system of generators of G. We suppose that the lines (OP_{m-1}) and (OP_m) bound the sector S. We consider a coordinates system in $\mathbb{R}^2$ in which $P_{m-1} = (1,0)$ and $P_m = (0,1)$. Then $\Lambda \otimes \mathbb{Q} = \mathbb{Q}^2$ and the coordinates of each P_i are positive and rational because $\{P_{m-1}, P_m\}$ is a basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Let $P = (x,y) \in S \cap \Lambda$ be any point, then there exists $(\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}^m$ such that:

$$P = \alpha_1 P_1 + \cdots + \alpha_m P_m.$$

For any $i = 1, \cdots, m-2$, there exists $\overline{\alpha_i}$, $0 \leq \overline{\alpha_i} < a_i$ such that: $\alpha_i \equiv \overline{\alpha_i} (\mod a_i)$. So, we can write P as :

$$P = \overline{\alpha_1} P_1 + \cdots + \overline{\alpha_{m_2}} P_{m-2} + \beta P_{m-1} + \gamma P_m,$$

with β, $\gamma \in \mathbb{Z}$. If the abscissa of P satisfies:

$$x(P) \geq X_0 := \max_{\{ 0 \leq \overline{\alpha_1} < a_1, \cdots, 0 \leq \overline{\alpha_{m-2}} < a_{m-2} \}} x(\overline{\alpha}_1 P_1 + \cdots + \overline{\alpha}_{m-2} P_{m-2})$$

then $\beta \geq 0$. If the ordinate of P satisfies:

$$y(P) \geq Y_0 := \max_{\{ 0 \leq \overline{\alpha_1} \leq a_1, \cdots, 0 \leq \overline{\alpha_{m-2}} \leq a_{m-2} \}} y(\overline{\alpha}_1 P_1 + \cdots + \overline{\alpha}_{m-2} P_{m-2})$$

then $\gamma \geq 0$. We take $A = (X_0, Y_0)$. $\square$

Back to the proof of Theorem 7.

Proof. We consider the map:

$$\begin{array}{cccc} I : M_*(\Gamma; \{i\}) & \longrightarrow & \mathbb{N}^2 \\ f & \longrightarrow & (\frac{k(f)}{2}, \mathrm{ord_i(f)} + \frac{\mathrm{k(f)}}{2}), \end{array}$$

here $k(f)$ is the weight of f and $\mathrm{ord_i(f)}$ is the vanishing order of f at i. We define $I(f)$ as the point $(I_1(f), I_2(f))$, it is an invariant of f.

By Proposition 6, we have the property:

$$I(D_\phi(f)) = I(f) + (1, \beta),$$

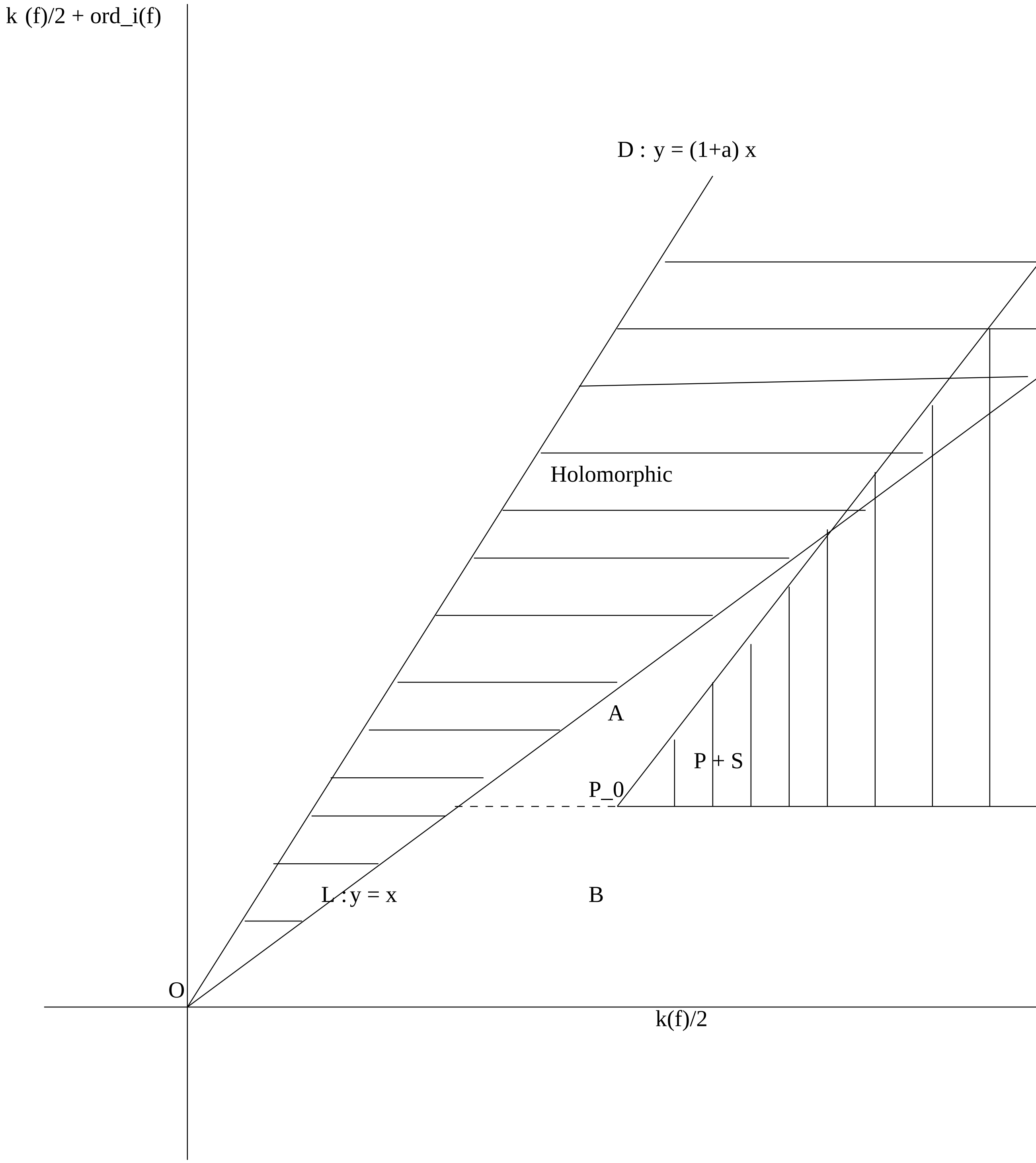
with $\beta \geq 0$. The case $\beta > 0$ may only happen if $I(f)$ is on the line $y = (2\mathcal{K} + 1)x$.

Let J be the ideal of modular forms of positive weights over Γ and $f_j, (j = 1, \cdots, d)$ be a basis of J/J^2. Let $R_0 = \langle f_1, \cdots, f_d, \omega, D_\phi(\omega) \rangle$, the ring generated by $f_1, \cdots, f_d, \omega$ and $D_\phi(\omega)$. We will construct a sequence of subrings of $M_*(\Gamma; \{i\})$:

$$R_0 \subset R_1 \subset \cdots \subset R_i \subset R_{i-1} \subset \cdots$$

such that for any i, R_i is finitely generated. We will proof that this sequence is stationary from a certain rank n_0, and also that $D_\phi(R_{n_0}) = R_{n_0} = R_{n_0+1}$.

We consider the semigroup of $\mathbb{N}^2$ that is finitely generated by the set $I(R_0) = \{I(f) | f \in R_0\}$. For $f \in M_*(\Gamma)$, we have $\mathrm{ord_i(f)} \leq \mathrm{a}\frac{\mathrm{k}}{2}$ for a certain $a > 0$, as a consequence of the formula of zeros for modular forms. If we choose a to be minimal, then the line D defined by $y = (a+1)x$ contains a non zero element of $I(R_0)$ and $I(R_0)$ has D as boundary. Hence, the semigroup $I(R_0)$ is embedded in the sector S delimitated by the lines (Ox) and D. The intersection of $I(R_0)$ and of the line (Ox) contains $I(\omega) = (2, 0)$ and $I(D_\phi(\omega)) = (3, 0)$, so this intersection contains all points $(a, 0)$ with $a \geq 3$. The set $I(R_0) \cap \{(x, y) | y \geq x\}$


k (f)/2 + ord_i(f)
D : y = (1+a) x
Holomorphic
A
P + S
P_0
L : y = x
B
O
k(f)/2

coincides with the set $I(M_*(\Gamma))$. Indeed, $\frac{k}{2} + \mathrm{ord}_i(f) \geq \frac{k}{2}$ if and only if $\mathrm{ord_i(f)} \geq 0$. Furthermore the elements of $M_*(\Gamma;\{i\})$ have no poles outside the $\Gamma-$orbit of i. It is clear that the group $I(R_0)$ is equal to $\mathbb{Z}^2$.

We apply the lemma to the semigroup $I(R_0)$, and deduce from this that there exists $P_0 \in I(R_0)$ such that $(P_0 + S) \cap \mathbb{Z}^2 \subset I(R_0)$, where S is the sector associated to $I(R_0)$ like in the Lemma.

We have the essential property that: if F is an element of $M_*(\Gamma;\{i\})$ and $I(F) \in (P_0 + S) \cap \mathbb{Z}^2$, then $F \in R_0$. Indeed there exists $g \in R_0$ such that $I(F) = I(g)$. So there exists a linear combination g_1 of F and g such that $I_1(g_1) = I_1(F)$ and $I_2(g_1) > I_2(F)$. By reiterating this construction, we obtain a sequence of points $I(g_i)$ which for i large exceed the line $y = x$. By induction in the opposite direction, one deduces that $F \in R_0$. In particular, if $I(D_\phi(f_j)) \in (P_0 + S) \cap \mathbb{Z}^2$, then $D_\phi(f_j) \in R_0$.

There are two cases: either $I(f_j)$ is in the left of the sector $P_0 + S$ (region A of the diagram), and then we must add the necessary number of derivations of f_j to get in to the sector $P + S$. Or case f_j lies below the sector (region B of the diagram): $I_2(f_j) < I_2(P_0)$.

We define the set:

$$E(R_0) = \{y \mid \nexists x \in \mathbb{N}, (x, y) \in I(R_0)\}.$$

In other words, $E(R_0)$ is the set of horizontal lines not occupied by $I(R_0)$. We have $0 \notin E(R_0)$, because $I(\omega) \in (Ox)$.

There exists y_0 such that: if $x > y \geq y_0$ then $(x, y) \in I(R_0)$. There also exists x_0 such that if $y < y_0$ and $y \notin E(R_0)$ then $I(R_0) \subset \{(x, y) \mid x \geq x_0\}$.

By induction, we define a sequence of rings: $R_{j+1} = \langle R_j, D_\phi(R_j)\rangle$. We also define a sequence of sets $E(R_0) \supset E(R_1) \supset \cdots$, by:

$$E(R_j) = \{y \mid \nexists x \in \mathbb{N}, (x, y) \in I(R_j)\}.$$

There exists y_j such that if $x > y \geq y_j$ then $(x, y) \in I(R_j)$. There also exists x_j such that, if $y < y_j$ and $y \notin E(R_j)$ then $I(R_j) \subset \{(x, y) \mid x \geq x_j\}$. Then, we have:

$$\cdots \subset \cdots \subset E(R_{j+1}) \subset E(R_j) \subset \cdots \subset E(R_0).$$

The sequence of finite sets $E(R_j)$ decreases, hence it is stationary. Then, there exists $j_0 \in \mathbb{N}$ such that $E(R_{j_0}) = E(R_{j_0+1})$. (this means, there will be no new occupied lines).

As a consequence, the sequence of rings R_j is stationary from a certain rank $n_0 \geq j_0$ with $D_\phi(R_{n_0}) = R_{n_0} = R_{n_0+1}$. Indeed, let $h_0 \in R_{j_0}$, after applying a suitable power of D_ϕ (this power is bounded by certain integer N independent of the choice of h_0) we obtain $h_1 \in R_{n_0}$

such that $I_1(h_1) \geq x_{n_0}$ (it is clear that $I_2(h_1) \notin E(R_{n_0})$). Since there exists $h_2 \in R_{n_0}$ such that $I(h_2) = I(D_\phi(h_1))$, then there exists a linear combination of h_2 and $D_\phi(h_1)$, equal to an element $h_3 \in R_{n_0}$ such that its image under I is a point in the same vertical line as $I(D_\phi(h_1))$ and with strictly greater ordinate. By reiterating this construction, we obtain a sequence of elements (h_n) in R_{n_0}. For n enough large, we have $I_2(h_n) \geq y_{n_0}$ so $h_n \in R_{n_0}$. This proves that $D_\phi(R_{n_0}) = R_{n_0}$ and $R_{n_0+1} = R_{n_0}$.

□

Remark 10. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete cocompact subgroup, and let $R \subset M_*^{\mathrm{mer}}(\Gamma)$ be a ring closed under D_ϕ which contains ω. Then, the ring $\widetilde{R} = R[\phi]$ is closed under D. Indeed $D(f) = D_\phi f + k\phi f$ for $f \in R$ and $D(\phi) = \omega + \phi^2$.

Corollary. **Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete cocompact subgroup. Then there exists a finitely generated ring closed under D which contains the ring of quasimodular forms over Γ. One can take $\widetilde{R} = R[\phi]$ with ϕ and R such as in Theorem** 10.

Proof. It is clear that $\widetilde{R}$ contains the ring of modular forms $M_*(\Gamma)$. On the other hand, $\widetilde{R}$ is closed under D by the last remark. By using Theorem 2, we deduce from this that $\widetilde{R}$ contains the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms. □

Remark 11. The ring $\widetilde{R}$ is generated by a finite number of meromorphic modular forms of positive weights, so in each weight k, the $\mathbb{C}$ vectoriel space $\widetilde{R}_k$ is finite dimensional. Furthermore, we have $\dim \widetilde{R}_k = \mathrm{O}(k^2)$, i.e. the finitely generated ring $\widetilde{R}$ has the same order of growth as its non-finitely genertaed subring $\widetilde{M}_*$.

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